

An Accurate Theoretical Model for the Thin-Wire Circular Half-Loop Antenna

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Abstract—The conventional Fourier series analysis for the thin-wire circular transmitting loop, or its image equivalent the half-loop, uses a delta-function generator for excitation. This method of excitation introduces two problems: it does not correspond to any realizable method of feeding the antenna so an accurate comparison with measurement is not possible, and it produces a divergent series for the input susceptance. To overcome these problems, a new theoretical model is used for the antenna: a half-loop driven through an image plane by a coaxial transmission line, with a transverse electromagnetic (TEM) mode assumed in the aperture of the coaxial line. This model is solved in a manner that preserves the simplicity of the original Fourier series analysis; all coefficients are obtained as closed form expressions. Input admittances calculated from this new theoretical model are in excellent agreement with accurate measurements.

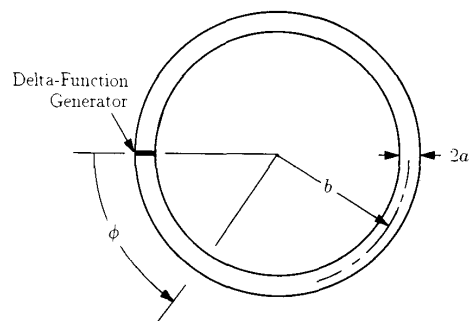


Fig. 1. Circular transmitting loop excited by delta-function generator.

I. INTRODUCTION

THE circular-loop antenna shown in Fig. 1 has radius b and is constructed from a perfectly conducting wire of radius a_i . When the wire is thin, $a_i \ll b$, $\beta_o a_i \ll 1$, the time-harmonic ($e^{j\omega t}$) surface current on the loop is predominantly in the azimuthal direction, ϕ , and is conveniently expressed as a Fourier series:

$$\vec{J}_s \approx \frac{I(\phi)}{2\pi a_i} \hat{\phi}$$

$$I(\phi) = I_0 + 2 \sum_{n=1}^{\infty} I_n \cos(n\phi). \quad (1)$$

Apparently the Fourier series was first used for the thin-wire circular loop by Pocklington in 1897 to study the scattering of an incident plane wave by a closed loop [1]. Later Hallén [2], Storer [3], and Wu [4] applied the Fourier series to the thin-wire circular *transmitting* loop; the excitation being a time-harmonic incident electric field concentrated at $\phi = 0$ on the surface of the loop:

$$E_\phi^i(\phi) = V_o \delta(\phi) / b, \quad (2)$$

the so-called "delta-function generator." A great accomplishment of this theory is that the Fourier series coefficients, I_n , for the transmitting loop are available as closed form

expressions [4]–[7]. In the formulation of Wu, they are

$$I_n = \frac{-jV_o}{\xi_o \pi} \frac{1}{a_n}, \quad n = 0, 1, 2, \dots \quad (3)$$

with

$$a_n = \frac{\beta_o b}{2} (K_{n+1} + K_{n-1}) - \frac{n^2}{\beta_o b} K_n, \quad (4)$$

and

$$K_n = \frac{1}{\pi} \mathcal{J}_0 \left[\frac{a_i}{b} \sqrt{n^2 - (\beta_o b)^2} \right] \mathcal{X}_0 \left[\frac{a_i}{b} \sqrt{n^2 - (\beta_o b)^2} \right] + \frac{1}{\pi} \left\{ \gamma + \ln \left[4(n^2 - \beta_o^2 b^2)^{1/2} \right] \right\} - \frac{2}{\pi} \sum_{m=0}^{n-1} \frac{1}{2m+1} - \frac{1}{2} \int_{x=0}^{2\beta_o b} [\Omega_{2n}(x) + jJ_{2n}(x)] dx. \quad (5)$$

Here $\mathcal{J}_0$ and $\mathcal{X}_0$ are the modified Bessel functions of the first and second kinds and order zero, Ω_{2n} is the Lommel-Weber function of order $2n$, J_{2n} is the Bessel function of the first kind and order $2n$, $\gamma = 0.57722 \dots$ is Euler's constant, and $\beta_o = 2\pi/\lambda_o = \omega/c$, $\xi_o = \sqrt{\mu_o/\epsilon_o}$.

The Fourier series analysis for the thin-wire circular loop antenna has been extended to treat several other antenna structures, these include: the loop with discrete or continuous impedance loading [8], [9], the loop with a spherical ferrite core [10], loops with concentric and eccentric spherical insulation in a dissipative medium [11]–[13], and the loop over a material half-space [14]–[17]. In all cases the delta-function generator is the excitation for the transmitting loop.

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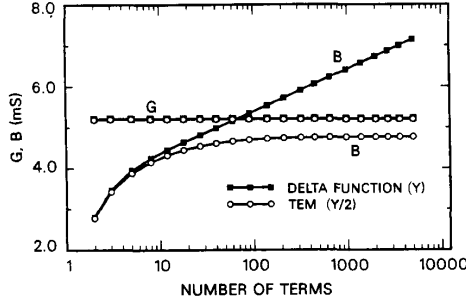


Fig. 2. Comparisons of input admittances as function of the number of terms in Fourier series for two antennas with different generators. $\beta_o b = 1.0$, $\Omega = 10.0$, $a_o/a_i = 2.30$ ($Z_0 = 50 \Omega$).

The use of the delta-function generator (2) leads to a well-known problem, the imaginary part of the series for the current (1) is divergent at the drive point: $\text{Im}[I(0)] \rightarrow \infty$ as $n \rightarrow \infty$. In terms of the input admittance

$$Y = G + jB = I(0)/V_o = \left(I_0 + 2 \sum_{n=1}^{\infty} I_n \right) / V_o, \quad (6)$$

this means that the series for the input susceptance B is divergent. This divergence is clearly shown in Fig. 2, where the admittance (G , B) is graphed as a function of the number of terms in the series for a loop with $\beta_o b = 1.0$ and $\Omega \equiv 2 \ln(2\pi b/a_i) = 10.0$. For practical calculations the series for the admittance (6) is usually truncated at 20 terms, and reasonable agreement is obtained with experimental values. Clearly this problem with the convergence detracts from the otherwise elegant theory for the thin-wire circular loop antenna.

To eliminate the divergent susceptance, Inagaki *et al.* replaced the delta-function generator by a pulse-function generator: a combination of two Heaviside unit-step functions that maintains a uniform electric field across a small gap in the conductor [18]. The problem with this model is that, like the delta-function generator, it does not correspond to any realizable method of excitation for the loop. One's initial thought may be that this model corresponds to a circular loop driven by a parallel-wire transmission line. However, the electric field between the conductors of a parallel-wire line is not uniform; and at the end of the line, the electric field is not rotationally symmetric about the axis of the loop conductor.

The half-loop antenna consists of a half circle of wire mounted vertically over a conducting image plane. The half loop is the image equivalent of the full loop. It can be analyzed using the procedure presented above when the exciting electric field is a delta-function or pulse-function generator applied between one end of the loop and the image plane. The current and input admittance of the half-loop antenna are then precisely twice those for the full-loop antenna (1), (6).

In this paper the aforementioned problems are eliminated by using a more realistic model for the excitation of the half-loop antenna: one that corresponds more closely to a practical method of excitation and does not lead to a divergent series for the susceptance. The Fourier series coeffi-

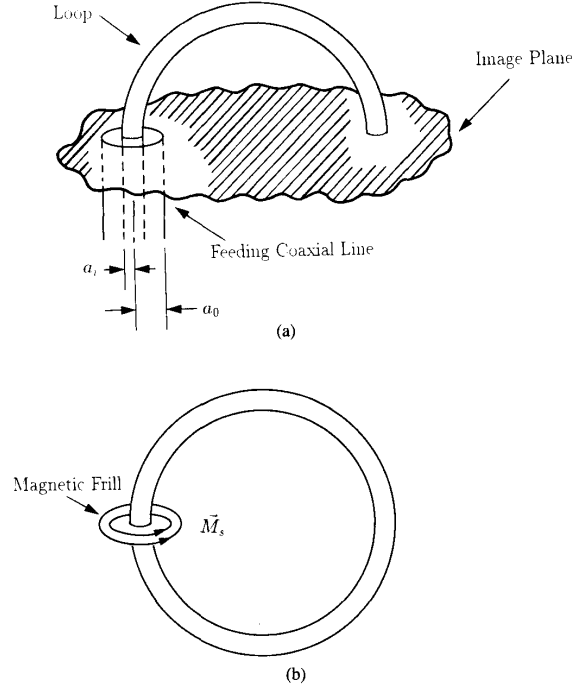


Fig. 3. (a) Half-loop driven through image plane from coaxial transmission line. (b) Equivalent full loop excited by magnetic frill.

cients for the current are given by closed-form expressions that are no more complicated than (3)–(5). Input admittances calculated from the new theoretical model are in excellent agreement with accurate measurements.

II. MODEL FOR THE ANTENNA

The new model for the antenna is a half-loop driven through a perfectly conducting image plane from a coaxial transmission line, as shown in Fig. 3(a). The electric field in the coaxial aperture is assumed to be that of a transverse electromagnetic (TEM) mode:

$$E_\rho(\rho) = V_o / [\rho \ln(a_o/a_i)], \quad a_i \leq \rho \leq a_o, \quad (7)$$

where ρ is the radial coordinate on the image plane, and a_i and a_o are the inner and outer radii of the coaxial line, respectively.¹ For electromagnetic analysis the geometry in Fig. 3(b) is equivalent to that in Fig. 3(a), [19]. Here a full-loop antenna is driven by a magnetic frill with the magnetic surface current density:

$$\vec{M}_s(\rho) = -2V_o / [\rho \ln(a_o/a_i)] \hat{\theta}, \quad a_i \leq \rho \leq a_o. \quad (8)$$

The electric field produced by this magnetic frill along the surface of the loop is the excitation for the antenna, $\vec{E}^i$. It is given by the following integral [20]:

$$\vec{E}^i = -\frac{1}{4\pi} \iint_S \vec{M}_s \times \nabla' \psi dS', \quad (9)$$

¹ The radius of the outer conductor a_o is chosen small enough that only the TEM mode propagates in the coaxial line.

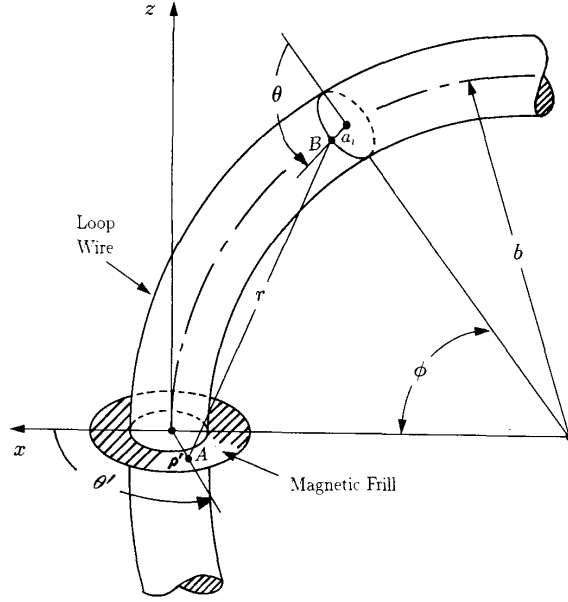


Fig. 4. Detail of geometry.

where

$$\psi \equiv \frac{e^{-j\beta_o r}}{r}, \quad (10)$$

and, referring to Fig. 4, r is the distance from the source point A (toroidal coordinates: $\rho', \theta', \phi' = 0$) on the frill to the observation point B ($\rho = a_i, \theta, \phi$) on the surface of the loop. The prime on ∇ indicates differentiation with respect to the coordinates of the source point, and the surface of integration is the area of the frill.

The field of the frill is most significant in the vicinity of the driving point, ϕ near $\phi = 0$. Due to the thin-wire assumption, $a_i \ll b$, the wire of the loop is nearly straight in this region, and the distance r can be approximated by

$$r \approx \sqrt{4b^2 \sin^2(\phi/2) + a_i^2 + \rho'^2 - 2a_i \rho' \cos(\chi)} \quad (11)$$

with $\chi = \theta - \theta'$. In addition, the component of the electric field tangential to the surface of the loop E_ϕ^i is approximately

$$E_\phi^i \approx E_z \cos \phi. \quad (12)$$

After inserting (8) into (9) and writing ∇' in terms of the coordinates (ρ', θ', z') , E_z is easily obtained, and (12)

$$b_n = \frac{b}{2\pi \ln(a_o/a_i)} \int_{\phi=-\pi}^{\pi} \int_{\chi=-\pi}^{\pi} \left\{ \frac{\exp[-j\beta_o \sqrt{4b^2 \sin^2(\phi/2) + 4a_i^2 \sin^2(\chi/2)}]}{\sqrt{4b^2 \sin^2(\phi/2) + 4a_i^2 \sin^2(\chi/2)}} - \frac{\exp[-j\beta_o \sqrt{4b^2 \sin^2(\phi/2) + a_i^2 + a_o^2 - 2a_i a_o \cos(\chi)}]}{\sqrt{4b^2 \sin^2(\phi/2) + a_i^2 + a_o^2 - 2a_i a_o \cos(\chi)}} \right\} \cos \phi \cos(n\phi) d\chi d\phi. \quad (19)$$

becomes

$$\begin{aligned} E_\phi^i(\phi) &= \frac{-V_o \cos \phi}{2\pi \ln(a_o/a_i)} \int_{\chi=-\pi}^{\pi} \int_{\rho=a_i}^{a_o} \frac{\partial \psi}{\partial \rho'} d\rho' d\chi \\ &= \frac{-V_o \cos \phi}{2\pi \ln(a_o/a_i)} \cdot \int_{\chi=-\pi}^{\pi} [\psi(\rho' = a_o) - \psi(\rho' = a_i)] d\chi. \end{aligned} \quad (13)$$

III. FOURIER SERIES SOLUTION

The electric field of the excitation $E_\phi^i(\phi)$ can be expressed as a Fourier series:

$$E_\phi^i(\phi) = \frac{V_o}{2\pi b} \left[b_o + 2 \sum_{n=1}^{\infty} b_n \cos(n\phi) \right], \quad (14)$$

with the coefficients given by

$$b_n = \frac{b}{V_o} \int_{\phi=-\pi}^{\pi} E_\phi^i(\phi) \cos(n\phi) d\phi. \quad (15)$$

The electric field produced by the current (1) in the loop $E_\phi^s(\phi)$ (the scattered field) also can be expressed as a Fourier series. This has been done in the previous formulations [4]–[7]:

$$E_\phi^s(\phi) = \frac{-j\zeta_o}{2b} \left[a_o I_o + 2 \sum_{n=1}^{\infty} a_n I_n \cos(n\phi) \right], \quad (16)$$

where the coefficients a_n are as in (4). When the boundary condition at the surface of the perfectly conducting loop is enforced:

$$E_\phi^i(\phi) + E_\phi^s(\phi) = 0, \quad (17)$$

and (14) and (16) are substituted, the Fourier series coefficients for the current are obtained:

$$I_n = \frac{-jV_o}{\zeta_o \pi} \left(\frac{b_n}{a_n} \right). \quad (18)$$

Notice that the Fourier series coefficients b_n (15) for the delta-function excitation (2) are simply $b_n = 1$, which with (18) leads to expression (3) for the Fourier series coefficients I_n for the current.

To obtain the current in the loop for the new method of excitation, the magnetic frill, the coefficients b_n first must be obtained. This requires evaluation of the integrals that result when (13) is substituted into (15)

The integration with respect to ϕ will be considered first; this can be split onto a number of integrals each of the form

$$\bar{K}_p(A) = \frac{1}{2\pi} \int_{\phi=-\pi}^{\pi} \frac{\exp \left[-j\beta_o b \sqrt{4 \sin^2(\phi/2) + A^2/b^2} + jp\phi \right]}{\sqrt{4 \sin^2(\phi/2) + A^2/b^2}} d\phi \quad (20)$$

where A^2/b^2 is a small quantity, such as $4(a_i/b^2) \sin^2(\chi/2) \ll 1$, and p is an integer. Wu evaluated similar integrals in obtaining the coefficients a_n [4]–[7], and his result is

$$\begin{aligned} \bar{K}_p(A) \approx & \frac{1}{\pi} \mathcal{K}_o \left\{ A \left[(p^2/b^2) - \beta_o^2 \right]^{1/2} \right\} \\ & + \frac{1}{\pi} \left\{ \gamma + \ln \left[4(p^2 - \beta_o^2 b^2)^{1/2} \right] \right\} \\ & - \frac{2}{\pi} \sum_{m=0}^{p-1} \frac{1}{2m+1} \\ & - \frac{1}{2} \int_{x=0}^{2\beta_o b} [\Omega_{2p}(x) + jJ_{2p}(x)] dx. \quad (21) \end{aligned}$$

Wu includes the restriction “at least for p not too large” with the evaluation; however, (21) can be shown to hold for general p [18], [21].

Only the term that involves $\mathcal{K}_o$ in (21) contains the factor A . Thus when (21) is used with (19), the subtraction in the integrand cancels all terms except those involving $\mathcal{K}_o$:

$$\begin{aligned} b_n = & \frac{1}{2\pi \ln(a_o/a_i)} \int_{\chi=-\pi}^{\pi} \\ & \cdot \left\{ \mathcal{K}_o \left[\frac{2a_i \sin(\chi/2)}{b} \sqrt{(n+1)^2 - (\beta_o b)^2} \right] \right. \\ & + \mathcal{K}_o \left[\frac{2a_i \sin(\chi/2)}{b} \sqrt{(n-1)^2 - (\beta_o b)^2} \right] \\ & - \mathcal{K}_o \left[\frac{\sqrt{a_i^2 + a_o^2 - 2a_i a_o \cos(\chi)}}{b} \right. \\ & \cdot \left. \sqrt{(n+1)^2 - (\beta_o b)^2} \right] \\ & \left. - \mathcal{K}_o \left[\frac{\sqrt{a_i^2 + a_o^2 - 2a_i a_o \cos(\chi)}}{b} \right. \right. \\ & \cdot \left. \left. \sqrt{(n-1)^2 - (\beta_o b)^2} \right] \right\} d\chi. \quad (22) \end{aligned}$$

Now the integrals that remain can be evaluated using the following definite integral [22]:

$$\begin{aligned} \mathcal{K}_o(c) \mathcal{K}_o(d) = & \frac{1}{2\pi} \\ & \cdot \int_{t=-\pi}^{\pi} \mathcal{K}_o \left[\sqrt{c^2 + d^2 - 2cd \cos(t)} \right] dt. \quad (23) \end{aligned}$$

The final result for the Fourier series coefficients for the electric field of the magnetic frill excitation is then

$$\begin{aligned} b_n = & \frac{1}{\ln(a_o/a_i)} \left(\mathcal{K}_o \left[\frac{a_i}{b} \sqrt{(n+1)^2 - (\beta_o b)^2} \right] \right. \\ & \cdot \left\{ \mathcal{K}_o \left[\frac{a_i}{b} \sqrt{(n+1)^2 - (\beta_o b)^2} \right] \right. \\ & \left. - \mathcal{K}_o \left[\frac{a_o}{b} \sqrt{(n+1)^2 - (\beta_o b)^2} \right] \right\} \\ & + \mathcal{K}_o \left[\frac{a_i}{b} \sqrt{(n-1)^2 - (\beta_o b)^2} \right] \\ & \cdot \left\{ \mathcal{K}_o \left[\frac{a_i}{b} \sqrt{(n-1)^2 - (\beta_o b)^2} \right] \right. \\ & \left. \left. - \mathcal{K}_o \left[\frac{a_o}{b} \sqrt{(n-1)^2 - (\beta_o b)^2} \right] \right\} \right). \quad (24) \end{aligned}$$

Notice that these coefficients are no more complicated than the coefficients a_n (4), (5); in fact, they are somewhat simpler since they do not involve the definite integrals of Ω_{2n} and J_{2n} .

IV. NUMERICAL RESULTS

With the Fourier series coefficients b_n for the TEM coaxial line excitation determined (24), the Fourier series coefficients I_n for the current in the half-loop (18) are readily obtained, and related quantities such as the input admittance (6) and radiated field [7] are easily calculated.

In Fig. 2 the input admittance for the half-loop with TEM excitation is graphed as a function of the number of terms in the Fourier series. The parameters for the loop are as before, $\beta_o b = 1.0$, $\Omega = 10.0$, and the dimensions of the coaxial line corresponds to a characteristic impedance $Z_o = 50 \Omega$, $a_o/a_i = 2.30 \dots$. The input conductance G of the loop is seen to be essentially the same as with the delta-function generator. However, the input susceptance B , unlike that for the delta-function generator, converges to a finite value as the number of terms in the Fourier series is increased.

The convergence of the series for the input admittance can be examined by considering the limit as $n \rightarrow \infty$ of the coefficients a_n and b_n . For a_n in the limit $n \rightarrow \infty$:

$$a_n \approx \frac{-n^2}{\beta_o b} K_n,$$

with the following components of K_n :

$$\int_{x=0}^{2\beta_o b} [\Omega_{2n}(x) + jJ_{2n}(x)] dx \approx 0,$$

$$\begin{aligned} \frac{1}{\pi} \left\{ \gamma + \ln \left[4(n^2 - \beta_o^2 b^2)^{1/2} \right] \right\} - \frac{2}{\pi} \sum_{m=0}^{n-1} \frac{1}{2m+1} \\ \approx -\frac{1}{24\pi n^2} + O\left(\frac{1}{n^4}\right), \end{aligned}$$

so, from (5):

$$K_n \approx \frac{1}{\pi} \mathcal{K}_o \left(\frac{a_i}{b} n \right) \mathcal{J}_o \left(\frac{a_i}{b} n \right) - \frac{1}{24 \pi n^2} + O \left(\frac{1}{n^4} \right) \\ \approx \frac{1}{2 \pi (a_i/b) n} + O \left(\frac{1}{n^2} \right);$$

thus,

$$a_n \approx -\frac{n}{2 \pi \beta_o a_i}. \quad (25)$$

Similarly, for b_n in the limit $n \rightarrow \infty$

$$b_n \approx \frac{2}{\ln(a_o/a_i)} \mathcal{J}_o \left(\frac{a_i}{b} n \right) \left[\mathcal{K}_o \left(\frac{a_i}{b} n \right) - \mathcal{K}_o \left(\frac{a_o}{b} n \right) \right] \\ \approx \frac{1}{\ln(a_o/a_i)(a_i/b)n} \\ \cdot \left[1 - \mathcal{K}_o \left(\frac{a_o}{b} n \right) / \mathcal{K}_o \left(\frac{a_i}{b} n \right) \right] \\ \approx \frac{1}{\ln(a_o/a_i)(a_i/b)n} \\ \cdot \left\{ 1 - \sqrt{\frac{a_o}{a_i}} \exp \left[- \left(\frac{a_o - a_i}{b} \right) n \right] \right\},$$

or for $n \gg b/(a_o - a_i)$,

$$b_n \approx \frac{1}{\ln(a_o/a_i)(a_i/b)n}. \quad (26)$$

The terms in the Fourier series for the input admittance (6), (18) are

$$2I_n/V_o = \frac{-j2}{\zeta_o \pi} \left(\frac{b_n}{a_n} \right), \quad n \geq 1, \quad (27)$$

which for $n \rightarrow \infty$ becomes

$$2I_n/V_o = \frac{j4\beta_o b}{\zeta_o \ln(a_o/a_i)} \left(\frac{1}{n^2} \right). \quad (28)$$

Now it is clear why the series for the input susceptance of the loop with TEM excitation converges: for large n it is proportional to

$$\sum \frac{1}{n^2},$$

a series which is known to converge. It is also clear why the series for the input susceptance of the loop with the delta-function excitation does not converge. For the delta-function excitation, $b_n = 1$ for all n , and (27) gives

$$2I_n/V_o = \frac{j4\beta_o a_i}{\zeta_o} \left(\frac{1}{n} \right). \quad (29)$$

Thus for large n the series for the input susceptance is proportional to

$$\sum \frac{1}{n},$$

a series which is known to diverge.

For practical calculations it is convenient to have an estimate of the error introduced when the Fourier series for the input admittance (6) is truncated at a finite number of terms N . Provided $N \gg b/(a_o - a_i)$, from (28) this error will be

$$\epsilon(N) \equiv Y(\text{infinite number of terms}) - Y(N \text{ terms}) \\ = \frac{-j2}{\zeta_o \pi} \sum_{n=N+1}^{\infty} \left(\frac{b_n}{a_n} \right) \approx \frac{j4\beta_o b}{\zeta_o \ln(a_o/a_i)} \sum_{n=N+1}^{\infty} \frac{1}{n^2}. \quad (30)$$

To estimate the last summation, notice that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \sum_{n=N+1}^{\infty} \frac{1}{n^2 - 1} = \frac{2N+1}{2N(N+1)} - \frac{1}{N},$$

so

$$\epsilon(N) \leq \frac{j4\beta_o b}{\zeta_o \ln(a_o/a_i)N}. \quad (31)$$

The TEM field (7) is not the exact field in the aperture of the coaxial line; higher order modes are also present. For coaxial apertures of practical size, the higher order modes have a small effect on the input admittance of the antenna. The difference between the exact admittance Y and the admittance computed with the TEM excitation Y_{TEM} ,

$$\delta Y = Y - Y_{\text{TEM}}, \quad (32)$$

is small and fairly independent of the type of wire antenna (linear, loop, etc.) excited by the aperture, provided the radius of curvature of the antenna wire near the aperture is large compared to the dimensions of the aperture. Morris determined δY for an infinitely long, linear antenna and presents his results in the form

$$\delta Y = \frac{j4\pi\beta_o a_i}{\zeta_o A}, \quad (33)$$

where A is a constant depending on the ratio a_i/a_o . [23]. Values of δY computed from his results are shown in Fig. 5. Notice that δY is small, a few tenths of a mS, for coaxial apertures of practical size ($Z_o \approx 50 \Omega$) and electrically thin wire antennas ($\beta_o a_i \ll 1$).

For the purpose of antenna design, it is convenient to have a graphical representation of the antenna input impedance $Z = 1/Y = R + jX$. In Fig. 6 the input impedance of the half-loop antenna is shown as a function of the electrical size of the loop $\beta_o b = 2\pi b/\lambda_o$. Curves are given for different values of the thickness parameter $\Omega = 2\ln(2\pi b/a_i)$. All results are for a driving coaxial line of characteristics impedance $Z_o = 50 \Omega$ and include the correction term δY (33); that is,

$$Y = Y_{\text{TEM}} + \delta Y \quad (34)$$

is graphed.

V. EXPERIMENTAL VERIFICATION

An experimental model was constructed for the half-loop antenna, and measurements of the input admittance of the loop were made to verify the theory. The half-loop antenna

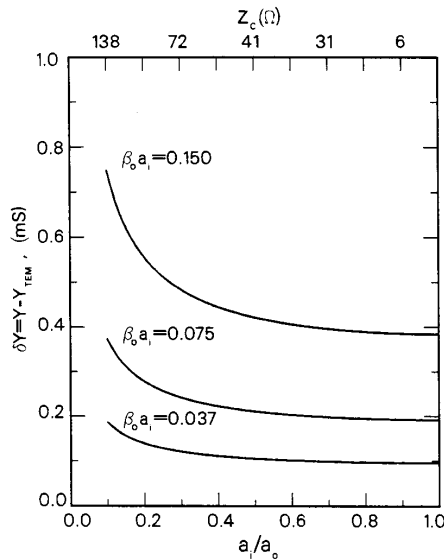


Fig. 5. Correction term δY versus the ratio of radii of inner and outer conductors of coaxial transmission line, a_i/a_o .

was mounted on an aluminum image plane (155 cm $\times$ 120 cm) and fed by a coaxial line from behind the plane. Absorbing material was placed around the front side of the image plane to reduce reflections from the support structure and edges of the plane. In addition, to further reduce errors due to reflections, the admittance was measured with the loop in eight different positions and these results averaged.

The method of feeding the half-loop is shown in Fig. 7. An APC-7 coaxial line feeds the antenna. A short spacer of length l_s and radius a_i is placed in the end of the line to change the characteristic impedance of the feeding line from 50 Ω (APC-7) to Z_o . The spacer and the loop wire have approximately the same radius. The spacer and loop wire were changed to obtain the three different values of characteristic impedance Z_o and thickness parameter Ω listed in Table I.

The reflection coefficient for each loop was measured at a reference plane in the APC-7 coaxial line, Fig. 7. The reflection coefficient was transformed through the length of line l to the step discontinuity in the inner conductor. The admittance was then calculated and the capacitance of the step discontinuity removed. Values of this capacitance were calculated using the program of Jurkus [24] and are listed in Table I. Finally, to obtain an experimental admittance for comparison with theory, the corrected admittance was transformed through the length of the l_s to the image plane.

The measurements were made over the frequency range 400 MHz $\leq f \leq 2$ GHz using a Hewlett-Packard model 8510A Network Analyzer with a Synthesized Sweeper. At each frequency point, 500 values of the reflection coefficient were measured and these averaged to reduce the effect of random noise.

Results for the three loops listed in Table I are shown in Figs. 8(a)–8(c) where both G and B are graphed as a

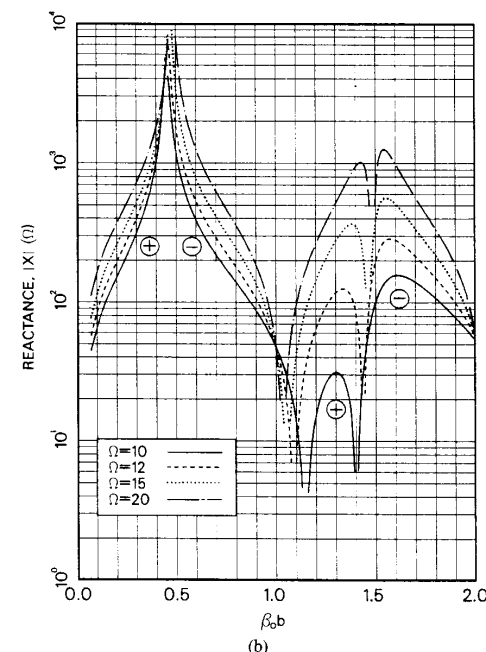
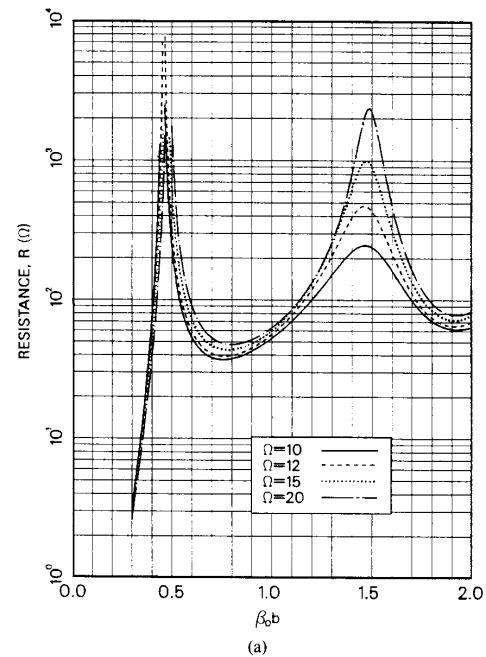


Fig. 6. Input impedance of half-loop antenna as function of electrical size $\beta_o b$. (a) Resistance (R). (b) Reactance (X).

function of the electrical size of the loop $\beta_o b$. The theory with the correction δY added (34) is shown as a solid line, while the experimental points are indicated by dots. The three cases correspond to feeding coaxial transmission lines of high ($Z_o \approx 86 \Omega$), nominal ($Z_o \approx 46 \Omega$), and low ($Z_o \approx 18 \Omega$) characteristic impedance. In all cases the theoretical and

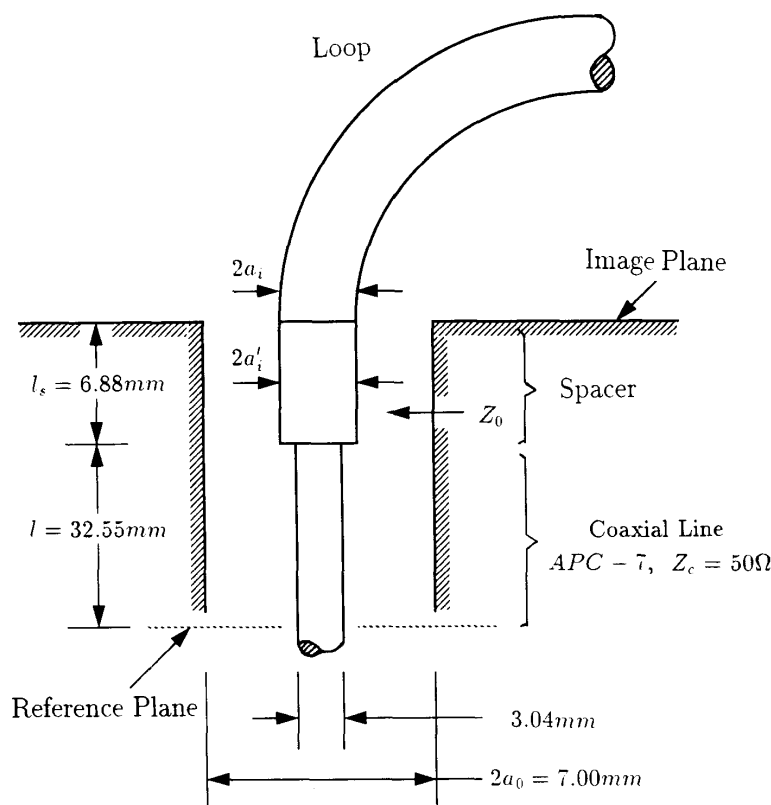


Fig. 7. Detail of method for feeding antenna in experiments.

TABLE I
PARAMETERS FOR THE HALF-LOOP ANTENNAS USED IN THE EXPERIMENTS

Loop	b (cm)	$2a_i$ (mm)	$\Omega = 2 \ln(2\pi b/a_i)$	$2a_i'$ (mm)	a_i/a_o	Z_0 (Ω)	Cap. of step (pF)
1	4.073	1.68	11.44	1.67	0.24	85.63	1.50×10^{-2}
2	4.063	3.25	10.11	3.25	0.46	46.04	8.06×10^{-2}
3	4.044	5.19	9.17	5.17	0.74	17.95	5.98×10^{-2}

experimental results are in excellent agreement. The slight difference that occurs in Fig. 8(c) for $\beta_o b \geq 1.4$ could be the result of the loop wire not meeting the thin-wire assumption of the theory. The thin-wire assumption requires $\beta_o a_i = 2\pi a_i/\lambda_o \ll 1$, the circumference of the loop wire must be small compared to the wavelength. At the highest frequency in Fig. 8(c) ($\beta_o b \approx 1.7$), the loop wire is over a tenth of a wavelength in circumference, $\beta_o a_i \approx 0.11$. Some of the discrepancy for this case may also be due to the difficulty in fabricating such a fat loop so that it is circular and the difficulty in centering the fat loop/spacer in the coaxial line; the spacing between the inner and outer conductors was only 0.9 mm.

Figs. 9(a)–9(c) are circle diagrams for the input admittances of the three half-loops: B is plotted against G with $\beta_o b$ as a parameter. These graphs emphasize the behavior near the resonance of the loop, $\beta_o b \approx 1.0$, and again show

the excellent agreement between the theory and the experiment.

VI. DISCUSSION AND CONCLUSION

The conventional Fourier series analysis for the thin-wire circular transmitting loop, or its image equivalent the half-loop, uses a delta-function generator for excitation. The delta-function generator introduces two problems into the analysis: it does not correspond to any realizable method of feeding the antenna so an accurate comparison with measurement is not possible, and it produces a divergent series for the input susceptance B . A new model for the antenna was introduced that overcomes both of these difficulties: a half-loop driven through an image plane by a coaxial transmission line, with a TEM mode assumed in the aperture of the coaxial line.

The Fourier series coefficients I_n for the current in the loop now involve two factors: the coefficients a_n (4) which appeared in the original analysis with the delta-function generator, and the new coefficients b_n (24);

$$I_n = \frac{-jV_o}{\zeta_o \pi} \left(\frac{b_n}{a_n} \right).$$

Both a_n and b_n are easily calculated. They involve the modified Bessel functions $\mathcal{K}_o$ and $\mathcal{J}_o$ for which computer programs are readily available. In addition, a_n involves

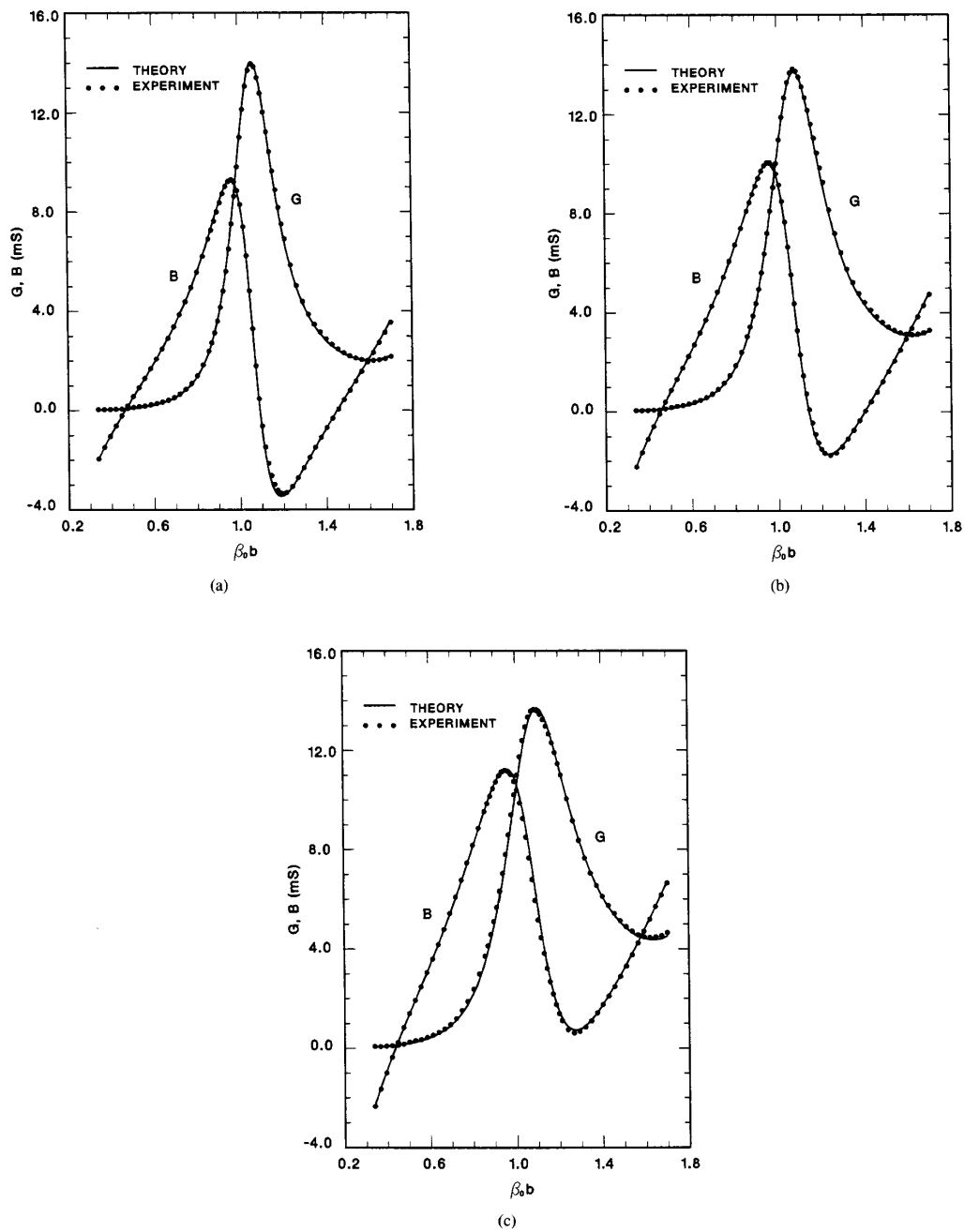


Fig. 8. Comparison of theoretical and measured input admittances (G , B) for half-loop antennas. (a) Loop 1, $\Omega = 11.44$; $Z_0 = 85.63 \Omega$. (b) Loop 2, $\Omega = 10.11$; $Z_0 = 46.04 \Omega$. (c) Loop 3, $\Omega = 9.17$; $Z_0 = 17.95 \Omega$.

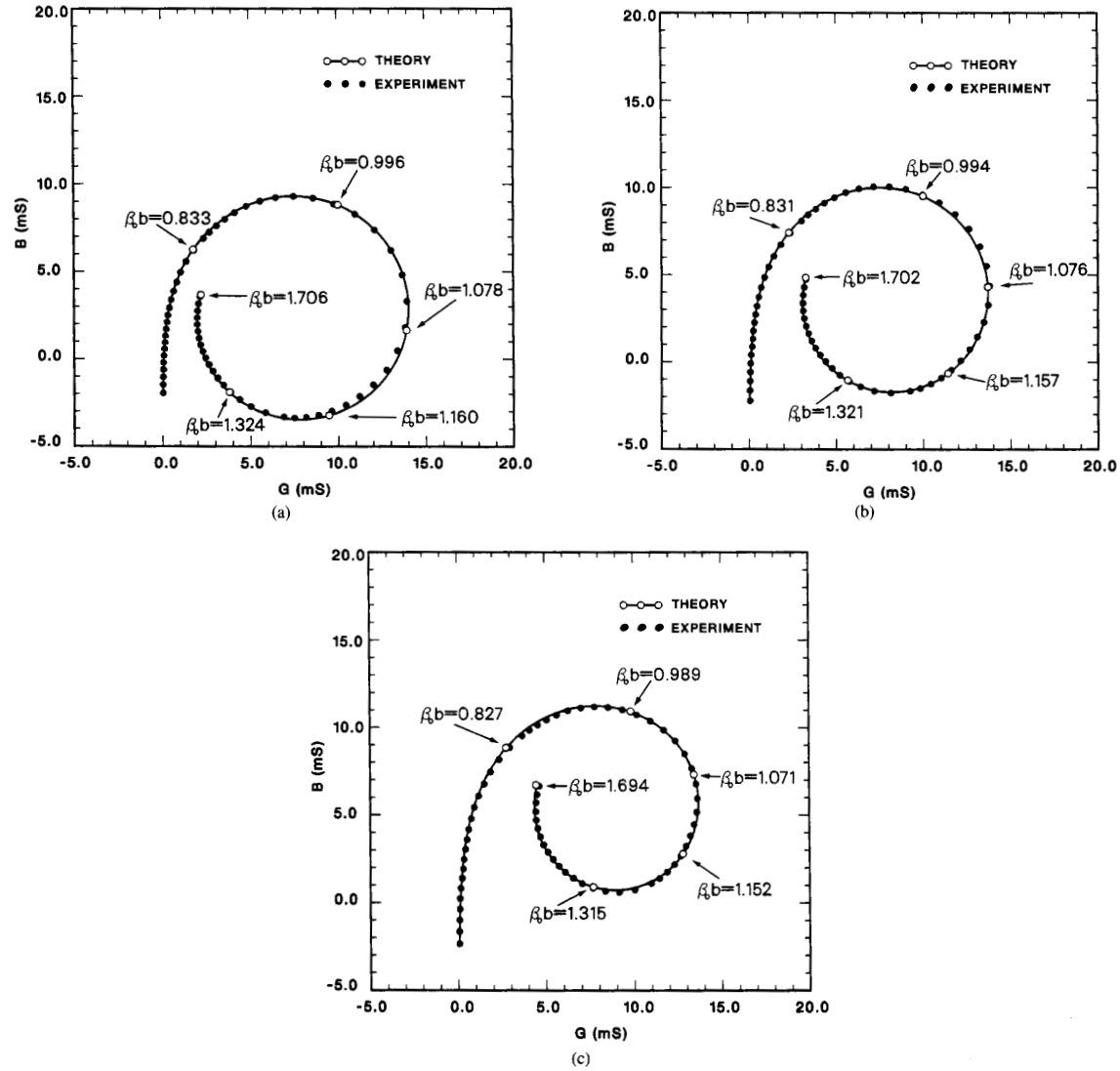


Fig. 9. Circle diagrams comparing the theoretical and measured input admittances for half-loop antennas. (a) Loop 1, $\Omega = 11.44$; $Z_0 = 85.63 \Omega$. (b) Loop 2, $\Omega = 10.11$; $Z_0 = 46.04 \Omega$. (c) Loop 3, $\Omega = 9.17$; $Z_0 = 17.95 \Omega$.

definite integrals of the special functions Ω_{2n} and J_{2n} which may appear formidable; however, these integrals can be evaluated using the simple, rapidly convergent series presented in the Appendix.

The aforementioned (Introduction) extensions of the Fourier series analysis to other antenna structures can easily be modified to include the new method of excitation. For example, the *full-loop antenna* with concentric or eccentric spherical insulation and delta-function excitation has Fourier series coefficients for the current of the form [11], [13]:

$$I_n = \frac{-jV_o}{\zeta_1 \pi (a_n + c_n)}, \quad (35)$$

where c_n accounts for the insulation (this coefficient is generally called b_n in the previous analyses). The Fourier series coefficients for the half-loop antenna with hemispherical insulation and TEM excitation are simply

$$I_n = \frac{-jV_o b_n}{\zeta_1 \pi (a_n + c_n)}, \quad (36)$$

where b_n is given by (24). In (35) V_o is the voltage of the delta-function generator; whereas, in (36) V_o is the voltage at the end of the coaxial transmission line.

The theory for the circular half-loop is now unique among the theories for thin-wire antennas; it provides simple, exper-

imentally verified, closed-form expressions (series) for the current and related quantities such as the input admittance and radiated field.

APPENDIX

The Lommel-Weber function has the series representation [25]:

$$\Omega_{2n}(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{x^{2k-1} (-1)^{k+1}}{[1^2 - (2n)^2][3^2 - (2n)^2] \cdots [(2k-1)^2 - (2n)^2]} \quad (37)$$

When this is substituted into the integral in (5) and integrated term by term, the following series is obtained:

$$\int_{x=0}^{2\beta_0 b} \Omega_{2n}(x) dx = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{[-(2\beta_0 b)^2]^k}{k[1^2 - (2n)^2][3^2 - (2n)^2] \cdots [(2k-1)^2 - (2n)^2]} \quad (38)$$

Now this result can be expressed as

$$\int_{x=0}^{2\beta_0 b} \Omega_{2n}(x) dx = -\frac{1}{\pi} \sum_{k=1}^{\infty} C_k, \quad (39)$$

where the recursive scheme

$$C_k = \frac{-(2\beta_0 b)^2(k-1)}{k[(2k-1)^2 - (2n)^2]} C_{k-1}, \quad (40)$$

$$C_1 = \frac{-(2\beta_0 b)^2}{1 - (2n)^2} \quad (41)$$

is used to compute the terms C_k .

A similar set of operations can be used to evaluate the integral in (5) of the Bessel function J_{2n} :

$$J_{2n}(x) = \left(\frac{x}{2}\right)^{2n} \frac{1}{\Gamma(2n+1)} \cdot \left[1 + \sum_{k=1}^{\infty} \frac{(-x^2/4)^k}{k!(2n+k) \cdots (2n+1)}\right], \quad (42)$$

$$\int_{x=0}^{2\beta_0 b} J_{2n}(x) dx = \frac{2(\beta_0 d)^{2n+1}}{\Gamma(2n+1)} \left(\frac{1}{2n+1} + \sum_{k=1}^{\infty} C_k\right), \quad (43)$$

where

$$C_k = \frac{-(\beta_0 b)^2(2n+2k-1)}{(2n+2k+1)k(2n+k)} C_{k-1} \quad (44)$$

$$C_0 = \frac{1}{2n+1}. \quad (45)$$

The series in (39) and (43) are convergent alternating series, and the errors made by neglecting all terms greater than the N th are easily estimated:

$$\left| \sum_{k=1}^{\infty} C_k - \sum_{k=1}^N C_k \right| \leq |C_{N+1}|.$$

Thus the absolute value of the relative error after summing N terms in either series is

$$\frac{\left| \sum_{k=1}^{\infty} C_k - \sum_{k=1}^N C_k \right|}{\left| \sum_{k=1}^N C_k \right|} \leq \frac{|C_{N+1}|}{\left| \sum_{k=1}^N C_k \right|}.$$

For loops of practical size, the series converge very rapidly; for example, with $\beta_0 b = 1.0$ both series have a relative error less than 10^{-10} when nine terms are used.

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